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Renormalisation of a diffusion on an anisotropic chain

M Napiórkowski

Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warszawa, Poland

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Abstract. A direct renormalisation method is used to calculate the index ν which characterises the mean square displacement of a walk on a one-dimensional, anisotropic, homogeneous lattice: $\langle R_n^2\rangle^{1/2} \sim n^{\nu}$ for large number of steps n. The two-parameter, exact analysis is carried out for arbitrary choice of the cell size. The results are shown to be independent of the cell size.

1. Introduction

The diffusion on random one-dimensional systems has recently attracted much interest (Derrida and Pomeau 1982, Alexander *et al* 1981). In this paper we consider the diffusion on the one-dimensional, homogeneous but anisotropic, infinite lattice. The diffusing particle can jump from the ith site either to the site $i-1$ with probability q or to the site $i+1$ with probability $1-q$. For large number of steps n the mean square displacement of the particle is described by the asymptotic law $\langle R_n^2\rangle^{1/2} \sim n^{\nu}$. Our purpose is to calculate the index ν by the direct renormalisation method. This method was previously applied to walks in one (Høye and Napiórkowski 1980) and in two dimensions (Shapiro 1978, Napiórkowski *et al* 1979).

Before applying the renormalisation scheme we solve our problem with the help of a simple equation. Let $P_k(n)$ denote the probability that the particle starting from the origin occupies after n steps the kth site. $P_k(n)$ obeys the equation

$$P_k(n+1) = qP_{k+1}(n) + (1-q)P_{k-1}(n). \tag{1}$$

The mean displacement $\langle R_n\rangle = \Sigma_k kP_k(n)$ and the mean square displacement $\langle R_n^2\rangle = \Sigma_k k^2 P_k(n)$ (after n steps) satisfy simple recurrence relations:

$$\langle R_{n+1}\rangle = \langle R_n\rangle + 1 - 2q \qquad \langle R_{n+1}^2\rangle = \langle R_n^2\rangle + 2\langle R_n\rangle(1-2q) + 1. \tag{2a, b}$$

Their solutions

$$\langle R_n\rangle = n(1-2q) \qquad \langle R_n^2\rangle = n^2(1-2q)^2 + n4q(1-q) \tag{3a, b}$$

imply the following formula for ν:

$$\nu = \begin{cases} 0.5 & \text{for } q = 0.5 \\ 1 & \text{for } q \neq 0.5. \end{cases} \tag{4}$$

The index ν depends in a discontinuous way on q. It takes on only two values: $\nu = 0.5$ (as for the random walk, $q = 0.5$) and $\nu = 1$ (as for the self-avoiding walk, $q = 0$ or $q = 1$).

2. Renormalisation

Now we shall obtain formula (4) by the renormalisation method. The basic idea is to formulate our walk problem as a critical problem which is then renormalised. The details of the procedure have been described elsewhere (Napiórkowski *et al* 1979, Høye and Napiórkowski 1980). Let us recall that the procedure consists of two main steps. First we collect groups of l ($l>1$) consecutive sites of a linear lattice into the sites of the renormalised lattice and then with each walk W on the original lattice we associate a unique walk W' on the renormalised lattice. This is done in a standard way: a renormalised site is visited during W' if and only if a majority of its l sites are visited during W. (If exactly one half of the sites are visited then an arbitrarily selected site must be visited during W to have the visit registered. We take this selected site to be the site number $\frac{1}{2}l+1$ in the cell.) Though in the one-dimensional case a nearest-neighbour site walk W generates only a nearest-neighbour cell walk W', we still need a two-parameter renormalisation. This is due to the anisotropy of the system: a step to the left is given a weight L and a step to the right is given a weight R. Next, by considering all the site walks compatible with a given cell step, one determines the renormalisation equations: $L'=F^L(L,R)$, $R'=F^R(L,R)$, the flow lines, the fixed points and the biggest relevant eigenvalue λ at each fixed point. This eigenvalue λ determines the exponent ν,

$$\nu=\ln l/\ln\lambda. \tag{5}$$

Consider a linear lattice divided into l-site groups, figure 1. In order to evaluate the first renormalisation equation $L'=F^L(L,R)$ we have to consider all the site walks that start at a certain site, called the s-site, and terminate at the corresponding site in the neighbouring cell to the left (this site is called the l-site) without visiting the corresponding selected sites in other cells. The l-site can be visited only once while the rest of the allowed sites ($2l-1$ of them) can be visited an arbitrary number of times. The second renormalisation equation is obtained by an analogous procedure in which the l-site is replaced by the r-site. Following previous work (Høye and Napiórkowski 1980) we introduce four functions: $g^L_{p,l}(n)$, $f^L_{p,l}(n)$, $g^R_{p,l}(n)$, $f^R_{p,l}(n)$. The $g^L_{p,l}(n)$ is the number of different n-step walks that end up l units to the left from the s-site and such that the left endpoint together with the p sites to the right of it are all available during the walk. The $f^L_{p,l}(n)$ is defined as $g^L_{p,l}(n)$ with the additional restriction that the l-site is visited only once. The functions $g^R_{p,l}(n)$ and $f^R_{p,l}(n)$ are defined analogously with the obvious interchange of the left and right directions. By symmetry $f^R_{p,l}(n)=f^L_{p,l}(n)=f_{p,l}(n)$ and $g^R_{p,l}(n)=g^L_{p,l}(n)=g_{p,l}(n)$. With the help of the auxiliary generating functions

$$F^L_{p,l}(L,R)=\sum_{n=0}^{\infty} f_{p,l}(n)L^{(n+l)/2}R^{(n-l)/2} \qquad G^L_{p,l}(L,R)=\sum_{n=0}^{\infty} g_{p,l}(n)L^{(n+l)/2}R^{(n-l)/2} \tag{6a,b}$$

$$F^R_{p,l}(L,R)=\sum_{n=0}^{\infty} f_{p,l}(n)L^{(n-l)/2}R^{(n+l)/2} \qquad G^R_{p,l}(L,R)=\sum_{n=0}^{\infty} g_{p,l}(n)L^{(n-l)/2}R^{(n+l)/2} \tag{6c,d}$$

the renormalisation equations take the obvious form

$$L'=F^L_{2l-1,l}(L,R) \qquad R'=F^R_{2l-1,l}(L,R). \tag{7a,b}$$

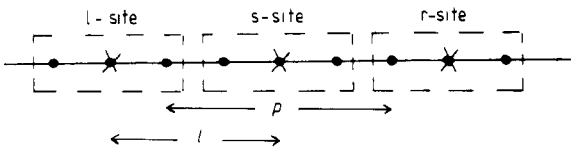

Figure 1. The partition of a lattice into l-site groups ($l = 3$, $p = 5$)

Whenever $l = 0$ we simplify the notation by putting $f_{p,0}(n) = f_p(n)$ and $g_{p,0}(n) = g_p(n)$ (by definition $f_p(0) = 0$, $g_p(0) = 1$). In this case we do not have to distinguish between the left and right directions in the generating functions because an equal number of steps is made in each direction:

$$F^L_{p,0}(L,R) = F^R_{p,0}(L,R) = F_p(L,R) \qquad G^L_{p,0}(L,R) = G^R_{p,0}(L,R) = G_p(L,R). \tag{8a,b}$$

Using the consequences of the definitions of $g_{p,l}(n)$ and $f_{p,l}(n)$

$$g_p(n) = 1 + f_p(n) + \sum_{n_1} f_p(n_1) f_p(n-n_1) + \sum_{n_1}\sum_{n_2} f_p(n_2) f_p(n_1-n_2) f_p(n-n_1) + \dots \tag{9a}$$

$$f_{p+1}(n+2) = g_p(n) \qquad f_{p+1,l+1}(n+1) = g_{p,l}(n) \tag{9b,c}$$

we obtain the following relations between the generating functions:

$$G_p(L,R) = (1 - F_p(L,R))^{-1} \tag{10a}$$

$$G^{L(R)}_{p,l}(L,R) = F^{L(R)}_{p,l}(L,R)/(1 - F_p(L,R)) \tag{10b}$$

$$F_{p+1}(L,R) = LRG_p(L,R) \tag{10c}$$

$$F^L_{p+1,l+1}(L,R) = LG^L_{p,l}(L,R) \tag{10d}$$

$$F^R_{p+1,l+1}(L,R) = RG^R_{p,l}(L,R) \tag{10e}$$

$$F^L_{p+2,l+1}(L,R) = LF^L_{p,l}(L,R)[G_{p+1}(L,R)G_p(L,R)/G_{p-l}(L,R)] \tag{10f}$$

$$F^R_{p+2,l+1}(L,R) = RF^R_{p,l}(L,R)[G_{p+1}(L,R)G_p(L,R)/G_{p-l}(L,R)]. \tag{10g}$$

Using (10a–g) and $F_1(L,R) = LR$ we are able to generate the renormalisation equations (7a, b) for arbitrary l.

Let us analyse the renormalisation equations in the simplest case $l = 2$. Later we shall prove that all the conclusions obtained in this case are valid for arbitrary l. Equations (7a, b) take the form

$$L' = L^2/(1-2LR) \qquad R' = R^2/(1-2LR). \tag{11a,b}$$

We look for the fixed points that satisfy the obvious (in our problem) condition $L + R = 1$. There are three of them with the corresponding eigenvalues and the values of the index ν calculated according to (5):

$$\begin{aligned} &L^*_{\mathrm{I}} = 0.5, R^*_{\mathrm{I}} = 0.5 && \lambda^{\mathrm{I}}_{1,2} = 4, 2 && \nu = 0.5 \\ &L^*_{\mathrm{II}} = 0, R^*_{\mathrm{II}} = 1 && \lambda^{\mathrm{II}}_{1,2} = 0, 2 && \nu = 1 \\ &L^*_{\mathrm{III}} = 1, R^*_{\mathrm{III}} = 0 && \lambda^{\mathrm{III}}_{1,2} = 2, 0 && \nu = 1. \end{aligned} \tag{12}$$

With the help of equations (11a, b) we check that the condition $L + R = 1$ implies $L' + R' = 1$; the $L + R = 1$ line is invariant under the renormalisation transformation.

The flow diagram is shown in figure 2. Our renormalisation group results are thus in complete agreement with the formula (4) obtained by a different method. The index ν is equal to 0.5 in the case of an isotropic system ($L = R = 0.5$) and is equal to 1 whenever the smallest anisotropy is present ($L = 1 - R \neq 0.5$).

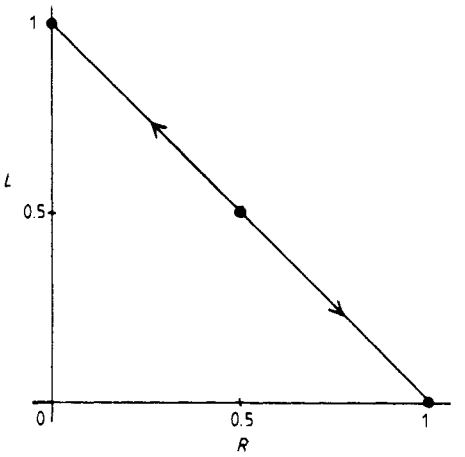

Figure 2. The flow diagram.

Now we shall prove by induction that our results remain valid for arbitrary l. First we show that if the renormalisation equations have the fixed points (12) for $l = k$ then they have the same fixed points for $l = k + 1$. This is checked with the help of equations (10*f*, *g*) and equation (13)

$$G_{p+1}(L, R) = (1 - LRG_p(L, R))^{-1} \tag{13}$$

obtained by combining (10*a*) and (10*c*). In the case of the fixed point $L = R = 0.5$ one proves by induction that $G_p(0.5, 0.5) = 2(p+1)/(p+2)$ which implies

$$G_{p+1}(0.5, 0.5)G_p(0.5, 0.5)/G_{p-l}(0.5, 0.5) = 2 \qquad \text{for } p = 2l - 1. \tag{14}$$

In the case of the remaining two fixed points we have

$$G_p(0, 1) = G_p(1, 0) = 1 \tag{15}$$

Equations (10*f*, *g*), (14), (15) indicate that the fixed points (12) are also present for $l = k + 1$.

Now we prove by induction that the renormalisation transformation (7*a*, *b*) linearised around these three fixed points

$$T_{L^*,R^*} = \begin{pmatrix} \left.\dfrac{\partial L'}{\partial L}\right|_{\substack{L=L^* \\ R=R^*}} & \left.\dfrac{\partial L'}{\partial R}\right|_{\substack{L=L^* \\ R=R^*}} \\ \left.\dfrac{\partial R'}{\partial L}\right|_{\substack{L=L^* \\ R=R^*}} & \left.\dfrac{\partial R'}{\partial R}\right|_{\substack{L=L^* \\ R=R^*}} \end{pmatrix} \tag{16}$$

has the form

$$T_{0.5,0.5} = \begin{pmatrix} \frac{1}{2}l(l+1) & \frac{1}{2}l(l-1) \\ \frac{1}{2}l(l-1) & \frac{1}{2}l(l+1) \end{pmatrix} \tag{17a}$$

$$T_{0,1}=\begin{pmatrix}0 & 0\\ l & l\end{pmatrix} \qquad T_{1,0}=\begin{pmatrix}l & l\\ 0 & 0\end{pmatrix}. \tag{17b,c}$$

Formulae (17*a–c*) are true for $l=2$ and if we assume that they hold for $l=k$ we can prove them for $l=k+1$ by differentiating both sides of (10*f*, *g*) with respect to L and R. In this calculus the following facts, which can be proved by induction for arbitrary l by differentiating both sides of (13) with respect to L and R, are used:

$$\left.\frac{\partial G_p}{\partial L}\right|_{\substack{L=0.5\\R=0.5}}=\left.\frac{\partial G_p}{\partial R}\right|_{\substack{L=0.5\\R=0.5}}=\frac{4}{3}\frac{p(p+1)}{p+2} \tag{18a}$$

$$\left.\frac{\partial G_p}{\partial L}\right|_{\substack{L=0\\R=1}}=1 \qquad \left.\frac{\partial G_p}{\partial R}\right|_{\substack{L=0\\R=1}}=0 \tag{18b}$$

$$\left.\frac{\partial G_p}{\partial L}\right|_{\substack{L=1\\R=0}}=0 \qquad \left.\frac{\partial G_p}{\partial R}\right|_{\substack{L=1\\R=0}}=1. \tag{18c}$$

The eigenvalues at the fixed points have the following form:

$$\begin{aligned} &L_{\mathrm{I}}^{*}=0.5, R_{\mathrm{I}}^{*}=0.5 && \lambda_{1,2}^{\mathrm{I}}=l^2, l && \nu=0.5\\ &L_{\mathrm{II}}^{*}=0, R_{\mathrm{II}}^{*}=1 && \lambda_{1,2}^{\mathrm{II}}=0, l && \nu=1\\ &L_{\mathrm{III}}^{*}=1, R_{\mathrm{III}}^{*}=0 && \lambda_{1,2}^{\mathrm{III}}=l, 0 && \nu=1. \end{aligned} \tag{19}$$

The last property we prove by induction is the invariance of the $L+R=1$ line under the renormalisation transformation. Once again we use equations (10*f*, *g*) to prove that for $L+R=1$

$$F_{2l-1,l}^{L}(L,R)=L^l/(L^l+R^l) \qquad F_{2l-1,l}^{R}(L,R)=R^l/(L^l+R^l). \tag{20a,b}$$

To see this we first check that for $L+R=1$

$$G_p=(L^{p+2}+R^{p+2}-LR^{p+1}-RL^{p+1})/(L^{p+3}+R^{p+3}-LR^{p+2}-RL^{p+2}). \tag{21}$$

Formula (21) is true for $p=0$ and can be proved by induction for arbitrary p with the help of (13). Equation (21) is then used to verify that for $L+R=1$

$$G_{2l}=1+LR[(L^l+R^l)/(L^{l+1}+R^{l+1})]G_{l-1}. \tag{22}$$

Using (22) and (10*f*) we get for $L+R=1$

$$\begin{aligned} F_{2l+1,l+1}^{L}(L,R)&=L\frac{L^l}{L^l+R^l}\frac{G_{2l}(L,R)G_{2l-1}(L,R)}{G_{l-1}(L,R)}\\ &=\frac{L^{l+1}}{L^{l+1}+R^{l+1}}\cdot\frac{G_{2l}(L,R)-1}{LRG_{l-1}(L,R)}=\frac{L^{l+1}}{L^{l+1}+R^{l+1}}. \end{aligned} \tag{23}$$

Equation (20*b*) is proved in an identical way. We thus see that if $L+R=1$ then

$$L'+R'=L^l/(L^l+R^l)+R^l/(L^l+R^l)=1. \tag{24}$$

3. Conclusions

The main result of this paper is an exact, two-parameter renormalisation group analysis of diffusion on an anisotropic chain. Our analysis is restricted in a natural way by the requirement that the sum of weights connected with the step to the left and the step to the right is equal to one. This condition reflects the normalisation of the probability distribution for stepping from a given site either to the left or to the right. We prove then, for arbitrary cell size, the discontinuous behaviour of the index ν as a function of the symmetry of the system: if the system is isotropic, $\nu = 0.5$; if not, $\nu = 1$. This is in agreement with the universality hypothesis: the symmetry of our system determines the universality class to which the system belongs.

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